Advanced Classical Mechanics

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1 Preamble

The document which follows is a study guide which I compiled in December of 2018 for the final exam of the Advanced Classical Mechanics course at Yale University. All mistakes or typos present in this document are my own. If you would like to correct any typos or add to the document, send me an email and I'd be happy to provide you with the raw files. Feel free to distribute this guide as you see fit. Use it well. - Hannah

2 Newtonian Dynamics and Kinematics

2.1 Newton's Laws and Their Galilean Invariance

Newton's Laws (which are invariant under Galilean Transformations)

1. A body remains at rest or moves with a uniform velocity unless acted upon bu an outside force.
2. $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$ where $\vec{p} = m\vec{v}$
3. Action-Reaction Law: $\vec{F_{12}} = -\vec{F_{21}}$

2.2 Single Particle Conservation Law

- Linear Momentum: If $\vec{F} = 0$, then $\dot{\vec{p}} = 0$ (linear momentum is conserved).
- Angular Momentum: If torque,$\vec{N}$, $\vec{N} = 0$ then $\dot{\vec{L}} = 0$ (angular momentum is conserved).
- Energy: If the force is conservative, $E = T + V = const$

Derivation For the Work-Kinetic Energy Theorem: The work done between points 1 and 2 is defined as $W_{12} = \int_1^2 F \cdot d\vec{s} = \int_1^2 m\dot{\vec{v}} \cdot d\vec{s}$. Applying that $d\vec{s} = \vec{v}dt$, we see this becomes $\int_1^2 m\dot{\vec{v}}(\vec{v}dt)$. This can be alternatively written as $\int_1^2 \frac{1}{2}m\frac{d}{dt}(\vec{v}^2)dt = \frac{1}{2}mv^2 = KE$

2.3 System of Particles: Conservation Laws

In general for a system of particles, the center of mass moves as if the mass of the system is at the center of mass.

- Linear Momentum: If external force $\vec{F^e} = 0$, then $\vec{p} =$ constant. ($\vec{F^e} = \dot{\vec{p}}$)
- Angular Momentum: If external torque,$\vec{N^e}$, $\vec{N^e} = 0$ then $\dot{\vec{L}} = 0$. ($\vec{N^e} = \dot{\vec{L}}$)

- Energy: If both internal and external forces are conservative, then E = constant.

The general procedure for doing these sorts of problems is as follows

1. Decompose the force on particle i into internal and external forces.
2. Make cancellations/assumptions regarding the internal forces that results in a conservation law.

Conditions for Conservative Force

- W_{12} between any two points is path independent.
- Force can be written as the gradient of some scalar function of position. ($F = -\nabla V(r)$)

2.4 Accelerated Coordinate Systems

All Newton's laws are valid only in intertial frames - frames which are neither accelerating nor rotating. All rotations have a common fixed point which is the origin. We can think of two successive rotations as a combined rotation vector where the individual ones add. This also gives us the idea that infinitessimal rotations commute. (It doesn't matter which order they are performed in, the result is still the same.)

- $(\frac{d\vec{A}}{dt})_{space} = (\frac{d\vec{A}}{dt})_{body} + \vec{\omega} \times \vec{A}$ for a generic vector $\vec{A}$
- $\frac{d\vec{\omega}}{dt}$ is the same in both frames as $(\vec{\omega} \times \vec{\omega}) = 0$
- $\vec{a}_{inertial} = \vec{a}_{body} + 2\vec{\omega} \times \vec{v}_{body} + (\frac{d\vec{\omega}}{dt})_{body} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$
- $\vec{v} = \vec{\omega} \times \vec{r}$
- $m\ddot{r} = F + 2m\dot{r} \times \omega + m(\omega \times r) \times \omega = F + F_{cor} + F_{cf}$

2.4.1 Motion on the Earth

To first order, say $\vec{r} = r_0 + r_1$ where r_0 is the zeroth order contributions (where $\vec{\omega} = 0$) and r_1 is the 1st order contributions.

- $\ddot{\vec{r_0}} = \vec{g}$ (solve for $r_0(t)$)
- $\vec{r_0}(t) = -\frac{1}{2}gt^2\hat{z}$ (now plug this in)
- to first order in ω
- $\ddot{\vec{r_1}} = -2\vec{\omega} \times \dot{\vec{r_0}}$
- $\ddot{\vec{r_1}} = 2gt\vec{\omega} \times \hat{z}$
- $r_1 = \frac{1}{3}gt^3\omega Sin(\theta)\hat{y}$

2.5 Particle on a Scale

$\vec{V}_e = \vec{a}_e = 0$. We know that the normal force will match the force of gravity such that $\vec{N} = -mg$. Recall the general force equation above

$$0 = F + 2m\dot{r} \times \omega + m(\omega \times r) \times \omega$$

Here our $F = \frac{-GmM_e\hat{r}}{\vec{r}^3} + \vec{N}$ and our Coriolis force is zero resulting in

$$0 = \frac{-GmM_e\hat{r}}{\vec{r}^3} + \vec{N} + m(\omega \times r) \times \omega = \frac{-GmM_e\hat{r}}{\vec{r}^3} - m\vec{g} + m(\omega \times r) \times \omega$$

This results in an effective acceleration of gravity as

$$\vec{g} = \frac{-GM_e\hat{r}}{\vec{r}^3} + (\omega \times r) \times \omega \quad (1)$$

where the second term depends on the colatitude.

2.5.1 Holonomic Constraints

Constraints which can be written as equations $f_j(X_1, X_2, ...X_{3N}, t) = 0$ j = 1,..... k. The net work done by the forces of constraint is zero. There are many difficulties with constraints are...

1. Not all constraints are independent
   - To address this, use generalized coordinates.
2. Forces of constraint are unknown (want a mechanism to get rid of forces of constraint)
   - eliminate forces of constraint from the formalism

3 Lagrangian Dynamics

Virtual displacements are defined as infinitesimal instantaneous displacement of the coordinates consistent with the constraints. We freeze the constraints at time t and consider the displacement δ_{x_i} consistent with these constraints at time t.

A very useful theorem is *D'Lambert's Principle*. This principle says that The net virtual work done by the forces of constraint is zero. $\sum f_i \delta_{x_i} = \sum(F_i^{(a)} - \dot{p}_i) = 0$. Where $F_i^{(a)}$ represents the applied force.The latter form of expression is superior as it gives the advantage that the forces of constraint have disappeared from the problem. This characterizes how the forces of constraint act during the motion of the particle. I will now provide an overview of how to arrive at the Euler-Lagrange equations from this principle. We begin with

$$\sum(F_i^{(a)} - \dot{p}_i) = 0$$

If all x_i are independent $\rightarrow F_i = \dot{p}_i$. We can therefore write

$$\sum[\sum(F_i^{(a)} - \dot{p}_i)\frac{\partial x_i}{\partial q_\lambda}]\delta q_\lambda = 0$$

We will define Q_λ as a generalized force conjugate to q_λ by

$$Q_\lambda = \sum F_i \frac{\partial x_i}{\partial q_\lambda}$$

We can make our equation simpler in terms of this generalized force Q_λ.

$$\sum[Q_\lambda - R_\lambda]\delta q_\lambda$$

But what is R_λ? Let's figure this out

$$\sum \dot{p}_i \frac{\partial x_i}{\partial q_\lambda} = \sum m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\lambda} = \sum m_i (\frac{d}{dt}\dot{x}_i)\frac{\partial x_i}{\partial q_\lambda} \tag{2}$$

With the chain rule we see that this is equivalent to

$$\sum m_i \frac{d}{dt}(\dot{x}_i \frac{\partial x_i}{\partial q_\lambda}) - \sum m_i \dot{x}_i \frac{d}{dt}(\frac{\partial x_i}{\partial q_\lambda})$$

Algebra shows that the second term is equivalent to $\frac{\partial T}{\partial q_\lambda}$, the first term is equivalent to $\frac{d}{dt}(\frac{\partial T}{\dot{q_\lambda}})$. Plugging this in for R_λ we see that

$$\sum[Q_\lambda - R_\lambda]\delta q_\lambda = \sum[Q_\lambda - (\frac{d}{dt}(\frac{\partial T}{\dot{q_\lambda}}) - \frac{\partial T}{\partial q_\lambda})]\delta q_\lambda$$

Now, using the fact that all q_λ are independent

$$(\frac{d}{dt}(\frac{\partial T}{\dot{q_\lambda}}) - \frac{\partial T}{\partial q_\lambda}) = 0.$$

If the forces are conservative we see that

$$Q_\lambda = \frac{-\partial V}{\partial q_\lambda}$$

Since V is velocity independent, this becomes the Euler-Lagrange equations which are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_\sigma}} - \frac{\partial L}{\partial q_\sigma} = 0$$

where $L = T - V$.

3.1 Example for Lagrangian Mechanics

3.2 Calculus of Variations

A single variational principle which implies the whole set of Lagrange equations. Let x be an independent variable defined over the interval $[x_1, x_2]$ and let y(x) be some differentiable function of x defined on this interval, such that $y'(x) = \frac{dy}{dx}$. Define $\phi = \phi[y(x), y'(x), x]$. In a two dimensional space

$$ds = [(dx)^2 + (dy)^2] = [1 + (y')^2]^{1/2}dx$$

define

$$\phi = [1 + (y')^2]^{1/2}$$

since ϕ has no explicit dependence on y, the Euler-Lagrange equation takes the form

$$\frac{d}{dx}\frac{y'}{[1 + (y')^2]^{1/2}} = 0$$

A trivial integration yields the equation for the straight line $y = mx + b$.We can also use the calculus of variations in order to prove Lagrange's equations from the calculus of variations. Begin with the action which we will call I.

$$\delta I = \delta \int L(q, \dot{q}, t) = \int \delta L(q, \dot{q}, t) = \int \sum[\frac{\partial L}{\partial q_\lambda}\delta q_\lambda + \frac{\partial L}{\partial \dot{q_\lambda}}\delta \dot{q_\lambda}]dt$$

Now in an aside we note that $\delta\dot{q_\lambda} = \frac{d}{dt}\delta q_\lambda$. Now, we use integration by parts to evaluate $\int \frac{\partial L}{\partial \dot{q_\lambda}}\delta\dot{q_\lambda}$. We have

$$\int \frac{\partial L}{\partial \dot{q_\lambda}}\delta\dot{q_\lambda} = -\int \frac{d}{dt}(\frac{\partial L}{\partial \dot{q_\lambda}})\delta q_\lambda dt + (\frac{\partial L}{\partial \dot{q_\lambda}})\delta q_\lambda|_{t_1}^{t_2} = -\int \frac{d}{dt}(\frac{\partial L}{\partial \dot{q_\lambda}})\delta q_\lambda dt$$

From Hamilton's principle of least action we know that $\delta I = 0$. Plugging this into our overall equation from above we see that

$$\delta I = \int \sum[\frac{\partial L}{\partial q_\lambda} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q_\lambda}})]\delta q_\lambda dt = 0$$

This gives us the Euler-Lagrange Equations as

$$\frac{\partial L}{\partial q_\lambda} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q_\lambda}}) = 0$$

3.3 Hamilton's Principle of Least Action

Hamilton's Principle of Least action says that the motion between t_1 and t_2 is such that the action is stationary for any $\delta q(t)$ with $\delta q(t_1) = \delta q(t_2) = 0$. In other words $\delta I = 0$ for any $\delta q(t)$ such that $\delta q(t_1) = \delta q(t_2) = 0$.

3.4 Method of Lagrange Multipliers

Lagrange multipliers are a particularly useful way of solving problems with forces of constraint. The book defines the general method of this as...

- Suppose not all q_λ are independent
- We have k constraints $f_j(q_1, ..., q_n, t) = 0$. Find an extremum of a function under the constraint.
- f(x,y,z) with constraint g(x,y,z) = 0 (method of Lagrange multipliers). We introduce a constraint λ and consider the extremum of $f - \lambda g$.

Trevor also came up with a step-by-step guide of this which is particularly useful.

1. Write down L in terms of necessary coordinates and unnecessary ones that will include the constraint. L$(q_1...., q_N, \dot{q_1}...\dot{q_N})$.
2. Write down the constraint equation: i.e. $q_2 = a$
3. Write down the derivative of that $\delta q_2 = 0$ (rearrange this to equal zero if needed)
4. Write $\delta I + \int_{t_1}^{t_2} \lambda(\text{whateverequalszero})dt = 0$. Do this for however many constraint equations.
5. Then write down $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_i}}) - \frac{\partial L}{\partial q_i}$ = (whatever multiplies $\delta q_i = 0$, 0 if it is not there)
   - These are the constraint equations.
   - This could also be viewed as the constraint force is the λ times (whatever equals 0) term written with δq_i
6. Then solve for whatever the problem is.
7. Consider a virtual displacement in the direction of the constraint, then write $\delta W = Q_{direction}\delta_{direction}$ (i.e. $Q_r \delta r$)
   - Remember sign, this often includes a negative.

3.4.1 One Cylinder Rolling on Another

See page 75 in Fetter and Walecka for this example. One uniform cylinder of mass m and radius R_2 slipping on another cylinder of radius R_1. We can write the kinetic energy as

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta_1}^2) + \frac{1}{2}(\frac{1}{2}mR_2^2)\dot{\theta}_2^2$$

Now consider the potential

$$V = mgrcos(\theta_1)$$

We now consider our equations of constraint, we have two of them

$$r = R_1 + R_2$$

the second is a little bit more complicated

$$R_1\theta_1 = R_2(\theta_2 - \theta_1)$$

Now take the variation of the first constraint and multiply by λ_1 and the variation of the second constraint and multiply that by λ_2. This yields

$$\lambda_1\delta r = 0$$

$$\lambda_1[(R_1 + R_2)\delta\theta_1 - R_2\delta\theta_2] = 0$$

Now use the return the Lagrangian, which is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta_1}^2) + \frac{1}{2}(\frac{1}{2}mR_2^2)\dot{\theta}_2^2 - mgrcos(\theta_1)$$

We then solve Lagrange's equations for the variables r, θ_1, θ_2 and we get the following

$$-mr\dot{\theta}_1^2 + mgcos(\theta_1) = \lambda_1$$

$$\frac{d}{dt}mr^2\dot{\theta}_1 - mgrsin\theta_1 = (R_1 + R_2)\lambda_2$$

$$\frac{1}{2}R_2^2\ddot{\theta}_2 = -R_2\lambda_2$$

Apply the constraints to the first of our Lagrange equations to get

$$-m(R_1 + R_2)\dot{\theta}_1^2 + mgcos(\theta_1) = \lambda_1$$

similarly, applying our constraint equations to the third Euler-Lagrange equations yields

$$-\frac{1}{2}m(R_1 + R_2)\ddot{\theta}_1 = \lambda_2$$

Now take the second and third Lagrange equations and set them equal using λ_2.

$$m(R_1 + R_2)^2\ddot{\theta}_1 - mg(R_1 + R_2)sin\theta_1 = (R_1 + R_2)\lambda_2 = -\frac{1}{2}m(R_1 + R_2)^2\ddot{\theta}_1$$

Simplifying this yields

$$\frac{3}{2}(R_1 + R_2)\ddot{\theta}_1 - gsin\theta_1 = 0$$

Multiply both sides by $\dot{\theta}_1$ and integrating we get

$$(R_1 + R_2)\dot{\theta}_1^2 = \frac{4}{3}g(1 - cos\theta_1)$$

Now, plug back into earlier equation to get

$$N(\theta_1) = \frac{1}{3}mg(7cos\theta_1 - 4)$$

3.5 Is h the Energy?

Define h as $h = \sum p_i \dot{q}_i - L$. This is numerically equivalent to the Hamiltonian, but it is not exactly the same. $h = h(q, \dot{q}, t)$. Is h the energy like the Hamiltonian? We will discuss this. The kinetic energy is given

$$T = \sum \frac{1}{2} m_i \dot{x}^2$$

where $x_i = x_i(q, t)$. So we see that $\dot{x}$ is given as

$$\dot{x_i} = \sum \frac{\partial x_i}{\partial q_\lambda} \dot{q_\lambda} + \frac{\partial x_i}{\partial t}$$

using the above, we can write

$$T = \sum \frac{1}{2} m_i (\sum \frac{\partial x_i}{\partial q_\lambda} \dot{q_\lambda} + \frac{\partial x_i}{\partial t})(\sum \frac{\partial x_i}{\partial q_\mu} \dot{q_\mu} + \frac{\partial x_i}{\partial t})$$

We see that this will result in $T = T_2 + T_1 + T_0$ where T_n is a homogenous equation of order n in $\dot{q}$. These terms are given by

$$T_2 = \frac{1}{2} m_i \frac{\partial x_i}{\partial q_\lambda} \frac{\partial x_i}{\partial q_\mu} \dot{q_\lambda} \dot{q_\mu}$$

$$T_1 = m_i \frac{\partial x_i}{\partial q_\lambda} \dot{q_\lambda} \dot{x_i}$$

$$T_0 = \frac{1}{2} m_i (\frac{\partial x_i}{\partial t})^2$$

The idea here is that we can write the Lagrangian in terms of these as $L = L_2 + L_1 + L_0$.

We also have **Euler's Theorem** which says if f is homogenous of degree n, then $\sum x_i \frac{\partial f}{\partial x_i} = nf$. We can apply this to these scenarios. Return to our definition of h.

$$h = \sum p_i \dot{q}_i - L$$

Using Hamilton's equations to simplify the first term

$$\sum p_i \dot{q}_i = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

use now that $L = L_2 + L_1 + L_0$ to split this up.

$$\sum p_i \dot{q}_i = \sum \dot{q}_i \frac{\partial L_2}{\partial \dot{q}_i} + \sum \dot{q}_i \frac{\partial L_1}{\partial \dot{q}_i} + \sum \dot{q}_i \frac{\partial L_0}{\partial \dot{q}_i}$$

now use Euler's theorem.

$$\sum p_i \dot{q}_i = 2L_2 + L_1 + 0L_0$$

now returning to h we see

$$h = 2L_2 + L_1 - (L_2 + L_1 + L_0) = L_2 - L_0$$

So now the question that we have to ask is when is this equal to E? We see that if the transformations do not depend on time, $x_i = x_i(q_1.q_2...)$ then only T_2 exists and the potential energy is velocity independent, then $L_2 = T_2 = T$. Since the potential is velocity independent, $L_0 = -V$. Therefore,

$$h = L_2 - L_0 = T + V = E$$

In conclusion, we have the following statements.

1. If the Lagrangian does not depend on time, then the hamiltonian is a constant of the motion. $\frac{\partial H}{\partial t} = 0$.
2. If there are only time independent potentials with time independent constraints, then the hamiltonian is not only a constant of the motion, but is also the total energy.

3.6 Symmetry Principles and Conserved Quantities: Noether's Theorem

If q_σ is a cyclic coordinate, then p_σ is constant.

- Noether's Theorem: There is a constant of the motion associated with a continuous symmetry.

We can prove this in the Lagrangian formalism. Consider a transformation near the identity.

$$q_i^{'} = q_i + \epsilon G$$

take the time derivative of this which is

$$\dot{q_i^{'}} = \dot{q_i} + \epsilon \sum \frac{\partial G}{\partial q_j} \dot{q_j}$$

so we know that

$$\delta \dot{q_j} = \epsilon \sum \frac{\partial G}{\partial q_j} \dot{q_j}$$

Now consider the variation in the Lagrangian.

$$\delta L = \sum \frac{\partial L}{\partial q_j} \delta q_j + \sum \frac{\partial L}{\partial \dot{q_j}} \delta \dot{q_j}$$

plug in $\delta \dot{q_j}$ from above.

$$\delta L = \sum \frac{\partial L}{\partial q_j} \epsilon G + \sum \frac{\partial L}{\partial \dot{q_j}} \epsilon \frac{\partial G}{\partial q_j} \dot{q_j}$$

Substitute to bring momenta into your equation

$$\delta L = \sum \dot{p_j} \epsilon G + \sum p_j \epsilon \frac{\partial G}{\partial q_j} \dot{q_j} = \epsilon \sum \frac{d}{dt}(p_j G_j)$$

For a conserved quantity, $\delta L = 0$, so $\frac{d}{dt}(p_j G_j) = 0$, so $\sum p_j G_j$ is conserved.

4 Two-body central force problem

4.1 The differential equation for the orbit

- $V_{eff} = V(r) + \frac{l^2}{2\mu r^2}$

We must make the following two assumptions about V.

1. V falls slower than $\frac{1}{r^2}$ when $r \to \inf$
2. V diverges slower than $\frac{1}{r^2}$ when $r \to 0$
3. Take V to be attractive.

We can classify orbits as the following.

1. For $E = V_{eff}(r_{min})$ the orbit is a circle at $r = r_{min}$.
2. For $V_{eff}(r_{min}) \leq E \leq 0$, the motion is bounded between the turning points r_1 and r_2.
3. For $E > 0$ only one turning point r_3 and the motion is unbounded.

Differential Equation for the Orbit

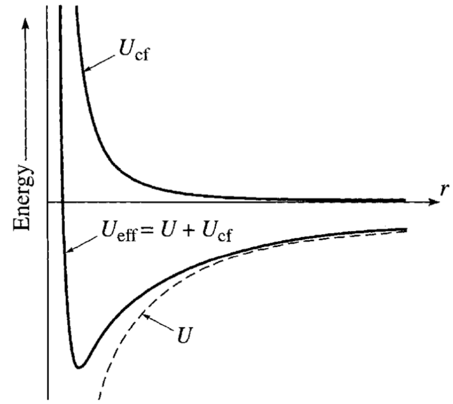


Figure 1: plot of effective potential

- $\frac{d^2u}{d\theta^2} + u(\theta) = -\frac{\mu}{l^2u^2(\theta)}F(r)$

We can derive this relatively simply. First, consider $r = r(\phi)$. Taking the time derivative we see that

$$\dot{r} = \frac{\partial r}{\partial \phi}\dot{\phi} = \frac{\partial r}{\partial \phi}\frac{l}{\mu r^2}$$

Now consider the equation for the energy, plug our $\dot{r}$ into this

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l}{2\mu r^2} + V = \frac{1}{2}\mu(\frac{\partial r}{\partial \phi}\frac{l}{\mu r^2})^2 + \frac{l}{2\mu r^2} + V$$

Now define a new quantity U such that $U = \frac{1}{r}$. Now

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi}$$

so the energy equation becomes

$$E = \frac{1}{2}\frac{l^2}{\mu}(\frac{du}{d\phi})^2 + \frac{l^2}{2\mu}u^2 + V$$

Differentiate both sides with respect to u

$$0 = \frac{1}{2}\frac{l^2}{\mu}2\frac{du}{d\phi}(\frac{du}{d\phi})^2\frac{d\phi}{du} + \frac{l^2}{\mu}u + \frac{dV}{du}$$

which gives the orbit equation

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{l^2}\frac{dV}{du} \tag{3}$$

Note that the solution to the differential orbit equation $u(\phi)$ has the property that if $u(\phi)$ is a solution, then $u(-\phi)$ will also be a solution. At a turning point $\frac{du}{d\phi} = 0$. So $u(\phi = 0) = u(-\phi = 0)$. Both solutions have the same initial conditions so the solutions are the same as each other (2nd order differential equation with the same initial conditions will have the same solution). so $u(\phi) = u(-\phi)$. In conclusion the orbit is symmetric about the turning point. Therefore, we choose the polar axis to go through the turning point.

Bertrand's Theorem: The only central forces that result in closed orbits for all bound particles are the inverse square law and Hooke's Law.

4.2 Kepler Problem

- $v = \frac{-k}{r} = -ku$
- $\frac{d^2u}{d\theta^2} + u(\theta) = \frac{\mu k}{l^2}$
- You end up getting the following equation for r which is the equation of a conic section with eccentricity e.
- $r = \frac{1}{\frac{\mu k}{l^2}(1+ecos(\theta))}$

Four possible cases for the eccentricity e.

1. For $E > 0, e > 1 \rightarrow$ hyperbola (scattering trajectory)
2. For E = 0, e = 1 $\rightarrow$ parabola (scattering trajectory)
3. For $\frac{-mk^2}{2l^2} < E < 0$, $0 < e < 1 \rightarrow$ ellipse (bound orbit)
4. For $E = \frac{-mk^2}{2l^2}$, e=0 $\rightarrow$ circle (bound orbit)

The Kepler Problem has 7 constants of the motion, 5 are independent. They are listed below.

1. Energy
2. Angular Momentum
3. Runge-Lenz Vector A (dependent on 2, $A\dot{L} = 0$
4. Center or Mass motion (3 different coordinates)
5. Motion is confined to a plane. (dependent on 4)

4.3 Derivation of the Virial Theorem

The fundamental equations of motion are

$$\dot{p_i} = F_i$$

We are interested in the quantity

$$G_i = \sum p_i \cdot r_i$$

Take the time derivative of this

$$\frac{dG}{dt} = \sum \dot{p_i} \cdot r_i + \sum \dot{r_i} \cdot p_i$$

substituting our fundamental equations of motion we have

$$\frac{dG}{dt} = \sum F_i \cdot r_i + \sum \dot{r_i} \cdot m\dot{r_i}$$

Simplifying the second term we see

$$\sum \dot{r_i} \cdot m\dot{r_i} = \sum mv_i^2 = 2T$$

this results in

$$\frac{dG}{dt} = \sum F_i \cdot r_i + 2T$$

now, we need to integrate. Multiply both sides by dt.

$$\int dG = \int (\sum F_i \cdot r_i + 2T)dt$$

Choose the integration bounds on the G term to represent one period. Therefore, the left hand side of this equation goes to zero.

$$0 = \overline{\sum F_i \cdot r_i} + \overline{2T}$$

Then we can solve to get the Virial Theorem

$$\overline{T} = -\frac{1}{2}\overline{\sum F_i \cdot r_i}$$

To get the more well known version of the virial theorem, substitute $-\frac{\partial V}{\partial r}$ for the force. Assume the potential follows $V = r^{n+1}$ choose $n = -2$ to get the well known result. This is because so many forces are inverse square law forces.

4.4 Runge-Lenz Vector

The main important thing about the Runge-Lenz Vector is that it is the fifth conserved quantity in the Kepler problem. We will show that it is conserved here. By Newton's second law we have

$$\vec{F} = \dot{\vec{p}} = -\frac{k}{r^2}\hat{r}$$

Now consider

$$\frac{d}{dt}(\vec{p} \times \vec{L}) = \dot{\vec{p}} \times \vec{L} + \vec{p} \times \dot{\vec{L}}$$

however the second term goes to zero as $\vec{L}$ is conserved. Now plug in $-\frac{k}{r^2}\hat{r}$ for $\dot{\vec{p}}$ and $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \mu\dot{\vec{r}}$. These simplifications yield

$$\frac{d}{dt}(\vec{p} \times \vec{L}) = -\frac{k}{r^2}\hat{r} \times (\vec{r} \times \mu\dot{\vec{r}}) = -\frac{\mu k}{r^2}\hat{r} \times (\vec{r} \times \dot{\vec{r}})$$

Use the vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

to simplify this to be

$$\frac{d}{dt}(\vec{p} \times \vec{L}) = -\frac{\mu k}{r^2}[(\hat{r} \cdot \dot{\vec{r}})\vec{r} - (\hat{r} \cdot \vec{r})\dot{\vec{r}}] = -\frac{\mu k}{r^2}[\dot{r}\vec{r} - r\dot{\vec{r}}] = \mu k \frac{d}{dt}[\frac{\vec{r}}{r}]$$

we see that

$$\frac{d}{dt}(\vec{p} \times \vec{L} - \mu k \frac{\vec{r}}{r}) = 0$$

Therefore the Runge-Lenz vector, given by

$$\vec{A} = \vec{p} \times \vec{L} - \mu k \frac{\vec{r}}{r}$$

is a constant of the motion. What is the direction of $\vec{A}$? We choose this to be the polar axis. Therefore it is in the direction of the turning point with magnitude $A = \mu k e$. Also worth noting is that this is an **over-integrable system**.

4.5 Classical Scattering

Assume we have a beam with intensity I that is colliding with the target. We want to ask the question: How many particles are scattered in each direction? This is calculating a differential cross section. In general this is given by

$$I = \frac{\text{number of particles}}{\text{(unit area perpendicular to beam)(unit time)}}$$

In general, this follows the image in the figure below. The impact parameter, which we will refer to as b (but is given by s in the figure), measures the distance from the axis. The particles are scattered into a solid spatial angle $d\Omega$ with change of impact parameter db. The general formula for

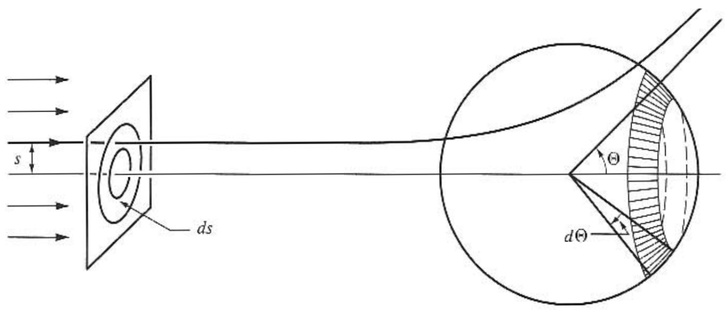


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

the differential cross section is given as

$$\sigma(\Omega)d\Omega = \frac{number\,of\,particles\,scattered\,into\,d\Omega}{(unit\,time)(I)}$$

The full explicit equation for the differential cross section goes as

$$\sigma = \frac{b}{sin\theta}\left|\frac{db}{d\theta}\right|$$

Some useful relations are $\theta = \theta(E, b)$,$l = \mu vb = b\sqrt{2mE}$ (as $E = \frac{1}{2}mv^2$). In general the process for solving these problems is as follows.

1. Given $\theta = \theta(E, b)$
2. Invert to find $b = b(\theta, E)$
3. This lets you find $\frac{db}{d\theta}$

However, sometimes it is not that easy to find $\theta = \theta(E, b)$. If you are not given this, use the following formula.

$$\theta(b) = \pi - \int_{r_m}^{\infty} \frac{dr}{r^2(\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}})}$$

A cool thing about scattering is that the procedures to describe scattering are the same in classical mechanics as quantum mechanics.

5 Small Oscillations

We will make the following assumptions about our system

1. Assume that we have n independent generalized coordinates $(q_1, ...q_N)$.
2. Assume that we have a conservative and time-independent potential $V = V(q_1, ...q_N)$
3. Assume that the constraints do not have any explicit time dependence.
4. Assume that the transformation equations are time-independent.

Lagrangian is time-independent so $\frac{d}{dt}(\frac{\partial T}{\partial \dot{q_\lambda}}) - \frac{\partial T}{\partial q_\lambda} = Q_\lambda = -\frac{\partial V}{\partial q_\lambda}$. We see here that the potential determines the generalized forces. Static equilibrium is characterized by the condition

$$\dot{q_\lambda} = \ddot{q_\lambda} = 0, \frac{\partial V}{\partial q_\lambda} = 0$$

So we are looking for the stationary points of V vs. all q_λ. For stable equilibrium, we are looking for local mins of V vs. all q_λ at the same time. At an equilibrium point, expand V around q_λ^0. Note that $\eta_\lambda = q_\lambda - q_\lambda^0$. Expanding this we see

$$V = V(q_\lambda) + \sum \frac{\partial V}{\partial r}|_0 \eta_\lambda + \frac{1}{2} \sum \frac{\partial^2 V}{\partial q_\mu \partial q_\lambda}|_0 \eta_\lambda \eta_\mu + ...$$

But this is a minimum so the first derivative term will be zero.

$$V = V(q_\lambda) + \frac{1}{2} \sum \frac{\partial^2 V}{\partial q_\mu \partial q_\lambda}|_0 \eta_\lambda \eta_\mu + ...$$

From our definition of η_λ we see that $\dot{\eta_\lambda} = \dot{q_\lambda}$. We can use this to construct the kinetic energy of the system

$$T = \frac{1}{2} \sum m_i (\frac{\partial x_i}{\partial q_\lambda} \frac{\partial x_i}{\partial q_\mu}) \dot{q_\lambda} \dot{q_\mu} = \frac{1}{2} m_{\lambda,\mu} \dot{q_\lambda} \dot{q_\mu}$$

where $m_{\lambda,\mu} = \sum m_i (\frac{\partial x_i}{\partial q_\lambda} \frac{\partial x_i}{\partial q_\mu})$. Using $\dot{\eta_\lambda} = \dot{q_\lambda}$ we see

$$T = \frac{1}{2} m_{\lambda,\mu} \dot{\eta_\lambda} \dot{\eta_\mu}$$

Now we can write out the Lagrangian as we have the kinetic and potential energies.

$$L = T - V = \frac{1}{2} \dot{\eta}^T M \dot{\eta} - \frac{1}{2} \eta^T V \eta$$

These are n coupled second order linear differential equations. We need to uncouple these equations. We do this by making a linear transformation to a set of coordinates where there are uncoupled. $\eta \to \zeta$ in general assume $\eta = R\zeta$ where R is a matrix of constants. Let's see how the Lagrangian transforms in these equations first, $\dot{\eta} = R\dot{\zeta}$, $\eta^T = \zeta^T R^T$, $\dot{\eta}^T = \dot{\zeta}^T R^T$. Using these, we redefine the Lagrangian as

$$L = \frac{1}{2} \dot{\zeta}^T (R^T M R) \dot{\zeta} - \frac{1}{2} \zeta^T (R^T V R) \zeta$$

Call $\tilde{V} = (R^T V R)$ and $\tilde{M} = R^T M R$. Then the Lagrangian becomes

$$L = \frac{1}{2} \dot{\zeta}^T \tilde{M} \dot{\zeta} - \frac{1}{2} \zeta^T \tilde{V} \zeta$$

Theorem: Assume that any M, V are real and symmetric matrices and M is positive definite (all eigenvalues are positive). Then there is a congruence transformation that diagonalizes M and V simultaneously i.e. there is a matrix R such that $R^T MR = I$ and $R^T VR$ is a diagonal matrix where the diagonal entries are the eigenvalues.
Proof of this Theorem: Since M is positive definite, so is M^{-1}. Therefore, $M^{-\frac{1}{2}} = \sqrt{M^{-1}}$ is a real matrix. There is matrix M such that M = A (diagonal matrix of γ's)A^{-1} where all γ's are positive. Therefore, we can also create a matrix $M^{-\frac{1}{2}} = A$(diagonal matrix of $\gamma^{1/2}$'s)A^{-1}. Now apply $M^{-1/2}$ as a congruence transformation. We get

$$M^{'} = (M^{-\frac{1}{2}})^T M(M^{-\frac{1}{2}}) = M^{-\frac{1}{2}} M M^{-\frac{1}{2}} = I$$

Then $V^{'} = M^{-\frac{1}{2}} V M^{-\frac{1}{2}}$ where $V^{'}$ is real and symmetric. This can be diagonalized by an orthogonal transformation O. Such that $O^T V^{'} O$ is diagonal. $O^T I O = O^{-1} O = I$. We will now define $R = M^{-1/2} O$, $R^T = O^T (M^{1/2})^T$.

The basic method for solving these problems is as follows.

1. Construct M(mass matrix), V(potential matrix)
2. Find the frequencies from $det(V - \omega^2 M) = 0$.
3. For each of the frequenciesω_j, find the column vector r^j from $(V - \omega^2 M) r^j$. You can then use these column vectors to construct the R matrix for $\eta = \zeta R$.
4. Transform to ζ coordinates using $\zeta = R^{-1}\eta$. (Note that when you do this varies from problem to problem. Usually there will be an indicator of when to do this.)
5. Diagonalize V and M using R.
6. Transform back to the original coordinates using $\eta = \zeta R$.

6 Special Relativity

6.1 The Lorentz Transformation

Galilean Transformations take the form of

$$\vec{r}^{'} = \vec{r} - \vec{v}t, \; t^{'} = t$$

In special relativity, we consider Lorentz transformations which take the form

$$\vec{r}^{'} = \vec{r} + (\vec{v} \cdot \vec{r}) \frac{\vec{v}}{v^2} (\gamma - 1) - \gamma \vec{v} t$$

$$t^{'} = \gamma (t - \frac{\vec{v} \cdot \vec{r}}{c^2})$$

Note that the postulates of special relativity require that speed of light and Maxwell's equations are the same in all reference frames. The quantity $c^2 dt^2 - (dr^2) = c^2 d\tau^2$ is also invariant under Lorentz transformations. (Where τ is the proper time.) We can define the mminkowski metric as follows.

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Be very careful, as this is defined differently in gravitational studies. The Minkowski metric is invariant under Lorentz transformations.

$$\eta = \Lambda^T \eta \Lambda$$

The matrix form of the Lorentz transformations is given as

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can also relate the time to the proper time which is the time measured in the particle's rest frame by $dt = \gamma d\tau$, where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$. Now that we have four dimensions (3 spatial and time) we need to create a new quantity for the velocity which we will call the 4 velocity. This is given as

$$U^\nu = \frac{dx^\nu}{d\tau} = (c\frac{dt}{d\tau}, \frac{dx^i}{dt}\frac{dt}{d\tau}) = (\gamma c, \gamma v^i)$$

where the four momentum is given as $p^\nu = mu^\nu$. Note that if $u^2 = c^2 > 0$ the vector is said to be timelike.

6.2 Relativisic Dynamics

Newtonian dynamics are not invariant under Lorentz transformations, so we must also express our force differently. The relativistic force is defined as

$$f^\nu = \frac{dp^\nu}{d\tau}$$

where the relativistic force is related to the usual force by

$$F^i = \frac{f^i}{\gamma}$$

Our energy and kinetic energy must also change. The Energy is defined as $E = \gamma mc^2$ which also obeys

$$E^2 = p^2c^2 + m^2c^4$$

where the kinetic energy is $T = (\gamma - 1)mc^2$.

6.3 The Lagrangain Formulation

This clearly requires that we redefine our lagrangian, which in special relativity is given as

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - V$$

in order to have h be the total energy under the conditions described earlier. We also have the covariant description (meaning it will hold in any Lorentz frame) which begins by having defining Lagrange's functions which are given by

$$\Lambda = \frac{1}{2}mu^\nu u_\nu$$

We can now define the Euler-Lagrange equations as

$$\frac{\partial}{\partial\tau}(\frac{\partial\Lambda}{\partial\dot{x_\nu}}) - \frac{\partial\Lambda}{\partial x_\nu} = 0$$

7 Rigid-Body Motion

A rigid body is a body for which $|\vec{r_{ij}}| = c_{ij} = const$ for any two points on the rigid body. There are 6 degrees of freedom here, 3 for the center of mass and 3 for the angles of orientation of the rigid body. We will now derive a few important equations in rigid body theory. The first being the kinetic energy. Begin by using the normal definition of kinetic energy while also using the fact that $\vec{v} = \vec{w} \times \vec{r}$.

$$T = \frac{1}{2}\sum m(\vec{\omega_\lambda} \times \vec{r_\lambda})^2 = \frac{1}{2}\sum m(\vec{\omega_\lambda} \times \vec{r_\lambda}) \cdot (\vec{\omega_\lambda} \times \vec{r_\lambda})$$

Now we will employ the identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C}) \cdot (\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D}) \cdot (\vec{B} \cdot \vec{C})$$

This simplifies the kinetic energy to

$$T = \frac{1}{2}\sum m[\omega_\lambda^2 r_\lambda^2 - (\vec{\omega_\lambda} \cdot \vec{r_\lambda}) \cdot (\vec{\omega_\lambda} \cdot \vec{r_\lambda})]$$

introduce another sum over i, j

$$T = \frac{1}{2}\sum\sum [m_\lambda \omega^2 r_\lambda^2 \delta_{ij} - m_\lambda X_{\lambda i} X_{\lambda j})\omega_i \omega_j$$

which simplifies using the definition of the moment of inertia tensor

$$T = \frac{1}{2}\sum I_{ij}\omega_i\omega_j = \frac{1}{2}\vec{\omega} \cdot I \cdot \vec{\omega}$$

Now moving onto the angular momentum. We begin by defining this in the usual way

$$\vec{L} = \sum \vec{r} \times \vec{p} = \sum m_\lambda (\vec{r_\lambda} \times \vec{v_\lambda})$$

now use $v = \omega \times r$

$$\vec{L} = \sum m_\lambda (\vec{r_\lambda} \times (\omega \times \vec{r_\lambda}))$$

using the BAC-CAB vector identity this becomes

$$L_i = \sum\sum m_\lambda [r_\lambda^2 \delta_{ij} - x_{\lambda i} x_{\lambda j}]\omega_j = \sum I_{ij}\omega_j$$

so $\vec{L} = I \cdot \vec{\omega}$.

7.1 Inertia Tensor, Principal Axis

Note that the moment of inertia is a second rank symmetric tensor that is constant in the body frame. We will now derive the parallel axis theorem. Consider a transformation of coordinates from $\vec{r} = \vec{r'} - \vec{a}$. As we want the moment of inertia in the original coordinates. Using our definition of the moment of inertia tensor we write

$$I_{ij} = \int d\vec{r}\rho(\vec{r})(r^2\delta_{ij} - x_i x_j)$$

where r' is the center of mass. Transform this to the center of mass coordinates

$$I_{ij} = \int d\vec{r'}\rho(\vec{r'})[(\vec{r'} - \vec{a})^2\delta_{ij} - (x_i' - a)(x_j' - a)]$$

expanding this we see

$$I_{ij} = \int d\vec{r}'\rho(\vec{r}')(r'^2\delta_{ij} - x_i'x_j') + \int d\vec{r}'\rho(a^2\delta_{ij} - a_ia_j) + \int d\vec{r}'\rho(-2\vec{r}'\cdot\vec{a} + a_ix_j' + a_jx_i')$$

The last term goes to zero and we are left with the parallel axis theorem

$$I_{ij} = I_{CM} + M(a^2\delta_{ij} - a_ia_j)$$

We can also define a *principal frame* which is a frame in which the moment of inertia tensor is diagonal.

7.2 Euler's Equations

Euler's equations combine the relation $\frac{d\vec{L}}{dt} = \vec{\tau}^e$ and $(\frac{dL}{dt})_{space} = (\frac{dL}{dt})_{body} + \vec{\omega}\times\vec{L}$. We begin by taking the principal frame to be the body frame noting that $L_i = I_i\omega_i$ where ω_i are the components of ω in the body frame.

$$(\frac{dL_i}{dt})_{body} + (\vec{\omega}\times\vec{L})_i = \tau_i^e$$

substitute in $L_i = I_i\omega_i$. Use i = 1 as an example.

$$I_1\frac{d\omega_1}{dt} + \omega_2L_3 - \omega_3L_2 = \tau_1^e$$

now substitute $L_3 = I_3\omega_3$ to get Euler's equations

$$I_1\frac{d\omega_1}{dt} = \omega_2\omega_3(I_2 - I_3) + \tau_1^e$$

the other two coordinates have Euler equations of

$$I_2\frac{d\omega_2}{dt} = \omega_3\omega_1(I_3 - I_1) + \tau_2^e$$

$$I_3\frac{d\omega_3}{dt} = \omega_2\omega_1(I_1 - I_2) + \tau_3^e$$

The orientation of the body frame is specified by a set of three angles these are Euler angles. $\hat{e_i^0}$ (intertial frame) $\rightarrow \hat{e_i}$ (body frame). This transformations is broken up into the product of three rotations which in summary are given by

1. Rotation by α around z.
2. Rotation by β around new y.
3. Rotation by γ around new Z.

8 Hamiltonian Dynamics and Transformation Theory

It is possible to recast the equations of motion so that they are invariant (in form) under a larger class of transformations. We will define the Hamiltonian as $H = H(q, p, t)$ where

$$H = \sum p_i\dot{q}_i - L$$

Where we now have twice the number of equations of motion. These are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

We can derive these from Hamilton's principle. Recall the original form of this was

$$\delta I = \delta \int L dt = 0$$

Now substituting in our definition for the the for the Hamiltonian

$$\delta \int (\sum p_i \dot{q}_i - H) = 0$$

Now vary the term in parentheses

$$\delta I = \int \sum (\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i) dt = 0$$

but we want the second term to have a $\dot{p}_i$ in it. Use integration by parts (with term canceling based on definition of t_1 and t_2.

$$\int dt \sum p_i \delta \dot{q}_i = \int \sum p_i \frac{d}{dt}(\delta q_i) dt = -\int \sum \dot{p}_i \delta q_i dt$$

Now plugging this into the above equation we see that

$$\delta I = \int \sum (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i) dt = 0$$

under this formulation q_i and p_i are independent so we see that we can split these up.

$$\int \sum (\dot{q}_i \delta p_i - \frac{\partial H}{\partial p_i}\delta p_i) dt$$

$$\int \sum (-p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i) dt = 0$$

Hamilton's equations naturally fall out of these.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

8.1 Canonical (Symplectic) Transformations

Define a matrix η such that

$$\eta = \begin{bmatrix} q \\ p \end{bmatrix}$$

Also define J such that is it a block matrix with 0's on the diagonal block and 1's and -1's on the upper and lower off diagonal blocks respectively. This matrix has the following properties

1. $J^2 = -I$
2. $JJ^T = 1$
3. $J^T = -J$

Hamilton's equations can be expressed in this formalism as

$$\dot{\eta} = J\frac{\partial H}{\partial \eta}$$

We will transform to a new set of coordinates such that $Q = Q(q, p, t)$ and $P = P(q, p, t)$. These will keep the form of Hamilton's equations, but there will be a new Hamiltonian K such that

$$\dot{Q} = \frac{\partial K}{\partial P}$$

$$\dot{P} = -\frac{\partial K}{\partial Q}$$

Where the new Hamiltonian K is defined by

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{\partial F}{\partial t}$$

where F is referred to as the generating function of the canonical transformation. We will discuss this further in the following section. We also have a matrix method to determine if the transformation is canonical. first consider the matrix M, which is defined as

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}$$

this is also called the *Jacobi Matrix*. We see that if the following equation holds, the transformation is canonical.

$$MJM^T = J$$

This is derived by

$$\dot{\zeta} = M\dot{\eta} = MJ\frac{\partial H}{\partial \eta} = MJM^T\frac{\partial H}{\partial \zeta}$$

8.2 Generating Functions

There are four different types of generating functions. Consider F_1 such that $F_1 = F_1(q, \dot{Q}, t)$. The above equation then becomes

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \sum \frac{\partial F}{\partial q_i}\dot{q}_i + \sum \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$$

Hence in order for this to hold we must have

$$p_i = \frac{\partial F_1}{\partial q_i}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}$$

Leaving

$$K = H + \frac{\partial F_1}{\partial t}$$

Consider $F_2 = F_2(q, P, t)$

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}$$

Now for $F_3 = F_3(p, Q, t)$

$$P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$q_i = -\frac{\partial F_3}{\partial p_i}$$

The fourth type is given by $F_4 = F_4(p, P, t)$. A canonical transformation in the vicinity of the identity generated by G is given by

$$\delta\eta = \epsilon J \frac{\partial G}{\partial \eta}$$

8.3 Poisson Brackets

The Poisson bracket is defined as

$$\Big\{\omega, \lambda\Big\} = \sum\Big(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{q_i}\Big)$$

This can be written more compactly in the symplectic notation as

$$\Big\{A, B\Big\} = \Big(\frac{\delta A}{\delta \eta}\Big)^T J \frac{\delta B}{\delta \eta}$$

The poisson bracket is invariant under a canonical transformation. Also if $\Big\{A, H\Big\}$, then A is a constant of the motion. We can also prove that if A and B are constants of the motion then $\Big\{A, B\Big\}$ is also a constant of the motion. This is proven using the Jacobi identity which says

$$\Big\{A, \Big\{B, C\Big\}\Big\} + \Big\{B, \Big\{C, A\Big\}\Big\} + \Big\{C, \Big\{A, B\Big\}\Big\} = 0$$

Substituting this in we see that the proof is relatively simple

$$\Big\{H, \Big\{A, B\Big\}\Big\} + \Big\{A, \Big\{B, H\Big\}\Big\} + \Big\{B, \Big\{A, H\Big\}\Big\} = \Big\{H, \Big\{A, B\Big\}\Big\} + 0 + 0 = \Big\{H, \Big\{A, B\Big\}\Big\} = 0$$

We now move on to **Liouville's Theorem** which says that the classical phase space distribution function is constant along the classical trajectory. Mathematically this is stated as

$$\frac{\partial \rho}{\partial t} = \Big\{H, \rho\Big\}$$

Proof of Liouville's Theorem: First, the volume in phase space is invariant under a canonical transformation.

$$d\zeta = det\Big(\frac{d\zeta}{d\eta}\Big)d\eta = det(M)d\eta = \pm d\eta$$

If the volume is preserved (and mass is preserved due to conservation of mass), then ρ will be constant. We also know that time evolution is a canonical transformation generated by H. We know for cannonical transformations that

$$\delta\eta = \epsilon J \frac{\partial G}{\partial \eta}$$

For q and p we have

$$\delta q_i = \epsilon \frac{\partial G}{\partial p_i}$$

$$\delta p_i = -\epsilon \frac{\partial G}{\partial q_i}$$

Hamilton's equations give how p and q evolve with time.

$$\dot{q_i} = \frac{\partial H}{\partial p_i}$$

$$\dot{p_i} = -\frac{\partial H}{\partial q_i}$$

Upon comparing these equations, it is perfectly clear that time evolution is a canonical transformation generated by H where $\epsilon = dt$. After we have shown that time evolution is canonical, we know $\rho(q(0), p(0), 0) = \rho(q(t), p(t), t)$. The use Ehrenfest's theorem to show that this translates to Liouville's theorem.

8.4 Symmetries and Conserved Quantities

Canonical transformations are given by

$$\delta A = \epsilon \Big\{ A, G \Big\}$$

For a continuous symmetry, $\delta H = 0$.

$$\delta H = \epsilon \Big\{ H, G \Big\} = 0 = \Big\{ H, G \Big\}$$

so the generator of a continuous symmetry is a constant of the motion.

8.5 Hamilton-Jacobi Theory

Consider a canonical transformation of the second type $F_2 = F_2(q, P, t)$ such that the new Hamiltonian K = 0. This means the new variables are all cyclic. This is called Hamilton's principal function defined by $F_2 = S(q, P, t)$. Hamilton-Jacobi Equations are given by

$$H(q, p = \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$$

If the Hamiltonian does not depend on time explicitly, we can write S as

$$S = w - \alpha t$$

where $W = W(q, \alpha)$ is Hamilton's characteristic function. Therefore, we can write

$$H(q, p = \frac{\partial W}{\partial q}) = \alpha_1$$

where α_1 is usually the energy. The other part of the transformation equations take the form of a new constant β. (Remember all of the new coordinates are cyclic.)

$$\frac{\partial S}{\partial \alpha} = \beta = \frac{\partial W}{\partial \alpha} - t$$

8.6 Action-Angle Variables

We will assume a system that is separable and periodic such that $w(q_1, ...q_N, \alpha_1, ...\alpha_N) = W_1(q_1, \alpha_1, ...\alpha_N) + W_2(q_2, \alpha_1, ...\alpha_N) ..... + W_N(q_N, \alpha_1, ...\alpha_N)$. We also assume periodicity in each pair (q,p). However, this periodicity takes two forms

1. Libration: q_i, p_i come back to the same point (ex: Harmonic Oscillator)
2. Rotation: p_i is a periodic function of q_i

We can then ask what are the frequencies ν_i? In order to determine this we will define *action variables*, the number of which is equivalent to the number of degrees of freedom. These are definted by

$$J_i = \oint p_i dq_i$$

These action variables yield $J_i = J_i(\alpha_1, ...\alpha_N)$. These can then be inverted in order to yield $\alpha_i = \alpha_i(J_1, ...J_N)$. These allow us to express the Hamiltonian in terms of action variables. In this way, we are replacing our momentum with J. The action variables are the new integration constant. We will now consider W as the generating function where the angle variables are the new coordinates in this transformation.

$$\theta_i = \frac{\partial W}{\partial J_i}$$

Then (θ_i, J_i) are the canonical coordinates and momenta generated by the generating function w. We can also define the frequencies as

$$\nu_i = \frac{\partial H}{\partial J}$$

these are such that $\beta_i = \theta_i - \nu_i t$. The proof to show that these are frequencies is relatively straightforward. Consider the system to undergo n_i periods τ_i such that $\Delta t = n_i \tau_i$ for each i. Therefore we have $\theta_i = \nu_i \Delta t = \nu_i n_i \tau_i$ we then have

$$d\theta_i = \sum \frac{\partial \theta_i}{\partial q_j} dq_j \tag{4}$$

recall that $\theta_i = \frac{\partial W}{\partial J_i}$ where $\theta_i = \theta_i(q, J)$. The above equation then becomes

$$d\theta_i = \frac{\partial}{\partial J}(\sum P_j dq_j)$$

Integrating this we see

$$\Delta\theta_i = \int d\theta_i = \frac{\partial}{\partial J_i} \int (\sum P_j dq_j) = \frac{\partial}{\partial J_i}(\sum n_j J_j) = n_j \delta_{ij} = n_i$$

this implies $\nu_i \tau_i = 1$. So indeed ν_i are frequencies.

9 Completely Integrable Systems

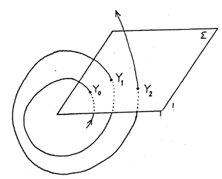
There exists a complete set of constants of the motion: $I_1, ..., I_n$ that are *involution* with each other, meaning that $\Big\{I_i, I_j\Big\} = 0 = \Big\{H, I_j\Big\} = \Big\{H, I_i\Big\}$. We can define action-angle variables

$$J_i = \oint_{\gamma_i} \sum P_j dq_j$$

where the γ_i's form a set of n-independent paths. We see that phase space is composed of invariant n-dimensional torii. Each torus is characterized by the given values of the constant $I_1, ...I_N$. If $n_1\nu_1 + n_2\nu_2 = 0$ with n_1, n_2 being integers, then the torus is a *resonating torus*. Otherwise the trajectory densely fills the torus. The question is can we look at completely integrable systems which are not separable? To check if a system is completely integrable, must find the second constant. One example of such a system is the Toda Hamiltonian.

10 Regular and Chaotic Motion of Hamiltonian Systems

In general most systems will not be completely integrable. The central question is how do we characterize the irregualrity of the dynamics? We are interested in the long term behaviour of the system, but we will do this by taking "snapshots" of the system called a *Poincare Surface of Section.* This would be the plane in the figure below. Call G the function that evolves your system with time.



$$G(\eta_i) = \eta_{i+1}$$

where G is symplectic. A fixed point η^* is a point that is mapped to itself $G(\eta^*) = \eta^*$. A *n-cycle* is a fixed point of G^n such that

$$G(\eta^0) = \eta_1, ...G(\eta_{n-1}) = \eta^0$$

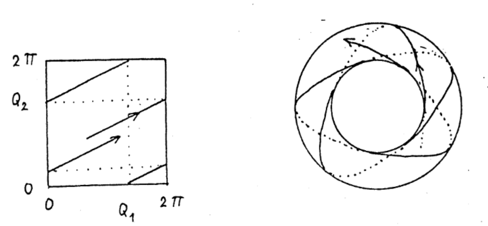
we can now define a matrix K such that

$$K = (\frac{\partial G}{\partial \eta})$$

Finding the eigenvalues and eigenvectors of this matrix will tell us how the system evolves. we see the following cases for the eigenvalues

1. If $\lambda > 1$ the system diverges from a fixed point η^*.
2. If $\lambda < 1$ the system converges to a fixed point η^*

The figure below characterizes a plot that we can make in the angle-variable space which then moves to a torus. Think of this like PacMan. (If you go off one side of the board, you will reappear on the other.) The *rotation number* is the ratio of the frequencies. $\frac{\nu_i}{\nu_n}$. If the rotation number is a rational



number, then every point on the invariant curve is an n-cycle. When we add a perturbation to the integrable system, then usually only two n-cycles survive - one is elliptic and the other is hyperbolic. We also have the **KAM Theorem** which states that there exists a slightly deformed torus with the same rotation number if the rotation number is "far enough" away from a rational number. This is defined as

$$\left|\alpha - \frac{m}{n}\right| > \epsilon n^{-5/2}$$

Where ϵ is a small number that depends on the perturbation. This means that for sufficiently small ϵ, most of the torii survive. How do we characterize chaotic vs. non-chaotic trajectories? This is done through separation of orbits. we can write this in terms of the action-angle variables as

$$J_i^{'} = J_i$$

$$\theta_i^{'} - \theta_i = (\nu_i^{'} - \nu_i)t + \beta_i^{'} - \beta_i$$

The general idea is that for regular trajectories, the distance is linear in time. However, for chaotic trajectories, the distance is exponential in time or

$$D(t) \sim e^{\lambda t}$$

where λ is the Lyapronov exponent. We have two cases of λ resulting in different trajectories.

1. When $\lambda = 0$ the trajectory is regular.
2. When $\lambda > 0$ the trajectory is chaotic